Robust Control of Servo DC Motor: LMI Approach

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Abstract: *The robust servomechanism problem consists in finding an LTI controller for the plant so that: (i) the resultant closed-loop system is asymptotically stable, (ii) asymptotic tracking occurs, and (iii) condition (ii) holds any arbitrary perturbation in the plant model (parametric uncertainty or dynamic uncertainty, including changes in model order) that do not cause the resultant closed-loop system to become unstable.*

Keyword: *Robust control, servo DC motor, Linear Matrix Inequality (LMI) approach, robust servomechanism problem.*

1. INTRODUCTION

The so-called servomechanism problem is one of the most basic problems to occur in the field of automatic control, and it arises in almost all application problems of the aerospace and process industries. In the servomechanism problem, it is desired to design a controller for a plant so that the outputs of the plant are independent, as much as possible, from disturbances which may affect the system (regulation occurs) and also such that the outputs asymptotically track any specified reference input signals applied to the system (tracking occurs), subject to the requirements of maintaining the closed-loop system stability. This paper examines some aspects of controller synthesis for the multivariable servomechanism problem when the plant to be controlled is subject to uncertainty. In this case, a controller is to be designed so that the desired regulation and the tracking take place in spite of the fact that the plant dynamics or/and parameters may vary by arbitrary, large amounts, subject only to the condition that the resultant closed-loop perturbed system remains stable. This problem is called the robust servomechanism problem.

The plant to be controlled is assumed to be described by the linear time invariant *LTI* model in state-space, with $x \in R^n$ is the state, $u \in R^{n_u}$ are the inputs that can be manipulated, $z \in R^{n_z}$ are the outputs that are to be regulated, $y \in R^{n_y}$ are the outputs that can be measured, $w \in R^{n_w}$ corresponds to the disturbances in the system (which in general cannot necessarily be measured) and $\varepsilon \in R^{n_\varepsilon}$ is the error in the system, which is the difference between the output y and the reference input signal y_{ref}, in which it is desired that the outputs y should track.

Two loop control structures are commonly used in motor drives, in speed applications and a third loop is added if positioning is required. This structure is widely perceived as very satisfactory, both from a control standpoint and from an apparatus protection point of view [7], [6]. The purpose of this paper is to quantify the sub optimal of a standard *PI* controller, nested in two-loop control design algorithm for a *DC* motor in speed servo applications with respect to robustness of the system uncertainties. Specifically, we will focus on the case of an uncertain moment of inertia. Our analysis and design tools will be the Linear Matrix Inequality *(LMI)* based methods, as developed in [2]. These methods are applicable to any motor and uncertainty type and are extendable to positioning problems as well.

As can be expected, our numerical experiments suggest that robust design procedures, such as the *LMI* method, offer potentially significant improvements in robust performance, in cases of large model uncertainties. Yet in low comparable cases, the *LMI* design method gives truly satisfactory dynamical response.

The paper re-examines the standard nested two-loop controller structure for a permanent magnet *DC* motor in speed servo applications. A robust synthesis, via Linear Matrix Inequalities *(LMI)* design, is compared with a conventional pole placed design. An analysis of a generic example demonstrates that robust design offers noticeable improvements in performance only in cases of relatively large model uncertainties.

2. MODELLING DC MOTOR

A common *actuator* in control systems is the *DC* motor. It directly provides rotary motion and coupled with wheels or drums and cables, can provide transitional motion. The motor torque, *T*, is related to the armature current, i_a, by a constant factor K_T. The back electromotive force *(e.m.f.)*, *e*, is related to the rotational velocity by the following equations:

$$T = K_T i_a,$$
$$e = K_e \omega.$$

The *DC* motor is provided by a pulse with modulated *(PWM)* voltage source in a majority of applications. A *PWM* power supply is an intrinsically non-linear device due to the switching mode of operation and saturation. At low frequencies the *PWM* unit is well approximated by a linear gain K_{PWM}, such an approximation being inaccurate at higher frequencies. Control gain roll off requirements are thus included in the conventional design algorithm. In order to make a meaningful comparison between the conventional and the robust controllers we require that control gains have comparable bounds at high frequencies. The control signal u_c is amplified by *PWM* amplifier to create the armature voltage u_a:

$$u_a = K_{PMW} u_c.$$

The *DC* motor equations based on Newton's law combined with Kirchhoff's law:

$$J\frac{d\omega}{dt} + B\omega = K_T i_a - T_L,$$
$$L_a \frac{di_a}{dt} + R_a i_a = u_a - K_e \omega$$

where:

$J\ [kg.m^2/s^2]$:	Moment of inertia of the rotor;
$B\ [N.m.s]$:	Damping ratio of the mechanical system;
$K = \left[\dfrac{V}{rad./\sec.}\right]$:	Back *e.m.f.constant*;
$K_T\ [N.m/A]$	Torque constant
$R[\Omega]$:	Electric resistance;
$L[H]$:	Electric inductance;
$i_a\ [A]$	The armature current;
$u_a\ [V]$:	The input (armature) voltage
$T_L\ [N.m]$	The load torque;
$\omega\left[\dfrac{rad.}{\sec.}\right]$:	The angular speed;

These equations can also be represented in state-space form. If we choose armature current and motor angular speed as our variables, we can write the equations as follows:

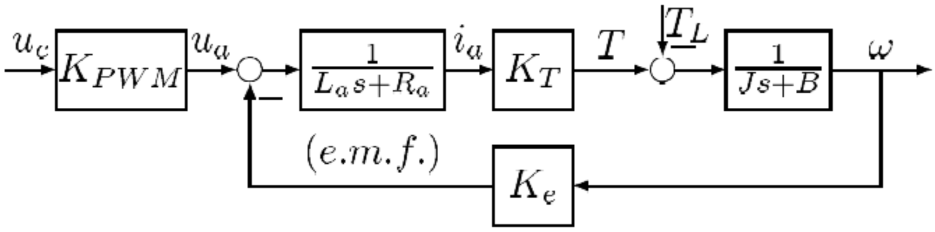

Figure 1. The block diagram of DC motor.

$$\frac{dx}{dt}=\frac{d}{dt}\begin{pmatrix} i_a \\ \omega \end{pmatrix}=\begin{pmatrix} -\frac{R_a}{L_a} & -\frac{K_c}{L_a} \\ \frac{K_T}{J} & -\frac{B}{J} \end{pmatrix}\begin{pmatrix} i_a \\ \omega \end{pmatrix}+\dots$$

$$\dots+\begin{pmatrix} \frac{K_{PWM}}{L_a} & 0 \\ 0 & -\frac{1}{J} \end{pmatrix}\begin{pmatrix} u_a \\ T_L \end{pmatrix},$$

$$y=\begin{pmatrix} i_a \\ \omega \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} i_a \\ \omega \end{pmatrix}.$$

The block diagram of DC motor it is shown in figure 1, [7].

We consider the following numerical values: $L_a=2.75E\text{-}6H$, $R_a=4\Omega$,$J=3.23E\text{-}6\ kg.m^2/s^2$, $B=3.51E\text{-}6\ N.m.s$, $K_T=0.0274\ N.m/A$ and $K_e=0.0274\ N.m/A$.

3. CONVENTIONAL CONTROLLERS

A simplified model of a servo *DC* motor is presented in figure 1. It is assumed that the measurements of the motor speed ω and the armature current i_a are available and that the control input is the voltage reference signal u_c, see figure 2.

A *PI* controller:

$$K_{PI}^{i_a}(s)=K_p^{i_a}+K_i^{i_a}\frac{1}{s},$$

for the current loop is designed in the first step. Here we ignore the back *e.m.f.* $E_a(s)=k_e\Omega(s)$, assuming that the current loop response dynamics is much faster. With this assumption, the closed loop current transfer function is:

$$G_{cl}^{i_a}\cong\frac{K_p^{i_a}K_{PWM}s+K_i^{i_a}K_{PWM}}{L_a s^2+\left(K_p^{i_a}K_{PWM}+R_a\right)s+K_i^{i_a}K_{PWM}}.$$

We also ignore the relatively small effects of $R_a i_a$ on $\frac{d}{dt}T_e\left(T_e=\frac{L_a}{R_a}\right)$. Under this assumption, the closed loop current transfer function is:

$$G_{cl}^{i_a}\cong\frac{K_p^{i_a}K_{PWM}s+K_i^{i_a}K_{PWM}}{L_a s^2+K_p^{i_a}K_{PWM}s+K_i^{i_a}K_{PWM}}.$$

We select the natural frequency $f_n^{i_a}=\frac{\omega_n^{i_a}}{2\pi}$ to be one tenth of the *PWM* switching frequency f_{PWM} (to ensure the validity of a linear approximation) and the damping $\zeta^{i_a}=1$ The coefficients $K_p^{i_a}$ and $K_i^{i_a}$ are then derived from the equations:

$$\frac{K_i^{i_a}K_{PWM}}{L_a}=\left(\omega_n^{i_a}\right)^2,\quad \frac{K_p^{i_a}K_{PWM}}{L_a}=2\zeta^{i_a}\omega_n^{i_a}.$$

The design of the *PI* speed controller:

$$K_{PI}^{\omega}(s)=K_p^{\omega}+K_i^{\omega}\frac{1}{s},$$

In the second step, a similar procedure is the following: the friction effects are

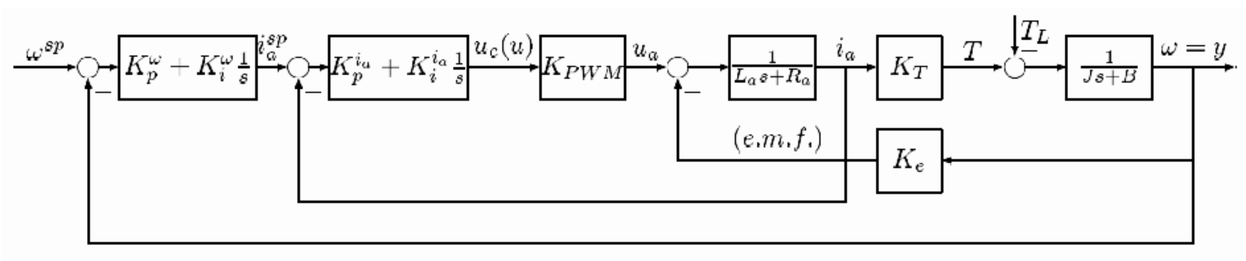

Figure 2. Block diagram of the conventional nested DC motor control structure.

neglected and the much faster current loop is approximated by the identity. The latter is justified by the selection of the speed closed loop natural frequency to be one tenth of that of the current loop. Under these assumptions, the closed loop angular speed transfer function is:

$$G_{cl}^{\omega} \cong \frac{K_p^{\omega} K_T s + K_i^{\omega} K_T}{L_a s^2 + K_p^{\omega} K_T s + K_i^{\omega} K_T}$$

Again a damping $\zeta^{\omega} = 1$ is fixed. The coefficients K_p^{ω} and K_i^{ω} are then derived from the equations:

$$\frac{K_i^{\omega} K_T}{J} = \left(\omega_n^{\omega}\right)^2, \quad \frac{K_p^{\omega} K_T}{J} = 2\zeta^{\omega}\omega_n^{\omega}$$

For our example we calculated the following data:

$$\omega_n^{i_a} = 0.924*10^3 \frac{rad.}{\sec.}, \ \omega_n^{\omega} = 0.924*10^2 \frac{rad.}{\sec.},$$

$$K_p^{i_a} = 0.0965, \ K_i^{i_a} = 45.467,$$

$$K_p^{\omega} = 0.4495, \ K_i^{\omega} = 21.1817.$$

Closed loop time responses are computed and depicted in figure 3, when the load inertia varies from *100%* to *200%* of its nominal value. We have assumed a constant load torque of $T_L=0.4[N.m]$ and speed reference tracking ±5%-variations from nominal value $\omega^{sp} = 1500[rpm] = 50\pi\left[\frac{rad.}{\sec.}\right]$.

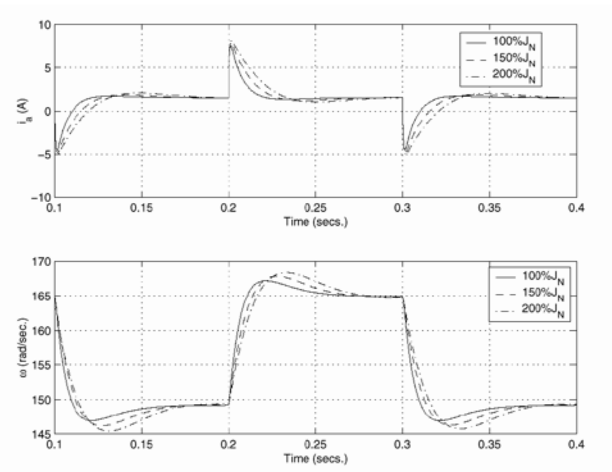

Figure 3. Conventional design: closed loop time response of the angular speed (ω) and current i_a, when load inertia varies from 100% to 200% of the nominal.

4. LMI CONTROLLER

Performance specifications (see [5]) for reference tracking and load rejection were based on the performance that was achieved in conventional design. The first step in the design procedure is to specify a generalized plant transfer matrix. The generalized plant *G* comprises the model of the original system and the various weighting functions that represent performance specifications. Plant uncertainty is represented by an unspecified block Δ with a known H^{∞} norm bound, interacting with *G* via disturbances signals ($w \in R^{n_w}$) and controlled signals that are to be regulated ($z \in R^{n_z}$). The generalized plant is driven by the exogenous multivariable inputs *w*, including disturbances, sensor noise and the tracking reference. A controlled outputs *z*, represents tracking errors, actuating commands and outputs that can be measured. The closed loop mapping $\mathrm{T_{zw}} : w \rightarrow z$ is required to be contractive. A stabilizing feedback controller, *K*, to be designed, will use the measured signal *y* and produces the control input *u*.

A speed loop *PI* controller and a current loop proportional controller ($K_i^{i_a} = 0$ in figure 2) comply with a:

a) closed loop robust stability for rotor inertia *J* varying from *100%* to *200%*;

b) reasonable speed tracking performance;

c) bounding of the command input u_C;

d) current limiting i_a.

The *LMI*-based controller design will be designed in three steps [3]: first, augmented the system by a Linear Fractional Representation *(LFR)*; then, deducing of Linear Matrix Inequalities *(LMI's)* that ensure the above specifications; finally, is showing of the results by numerical simulation.

4.1. A Construction of the LFR

The *LFR* model can be constructed systematically, starting by the system's dynamic equations (see [1]). It results in a model like the form depicted in figure 4, where the *G(s)* represents the *nominal* linear time-invariant system [4].

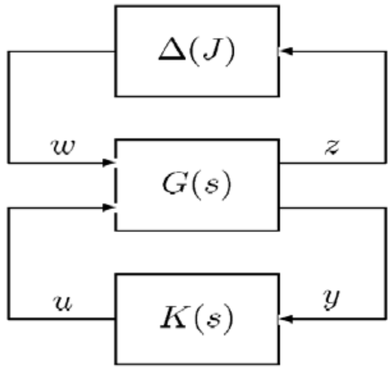

Figure 4. Linear Fractional Representation for uncertain non-linear systems.

The matrix $\Delta(J)$ contains rotor inertia uncertainties and is connected to the nominal system via input *w* and output *z*. We can write:

$$\frac{dx}{dt} = \begin{pmatrix} -\frac{R_a}{L_a} & -\frac{K_e}{L_a} & 0 \\ \frac{K_T}{J_N} & -\frac{B}{J_N} & 0 \\ 0 & -1 & 0 \end{pmatrix} x + \ldots$$

$$\ldots + \begin{pmatrix} 0 \\ -\frac{1}{J_N} \\ 0 \end{pmatrix} w + \begin{pmatrix} \frac{K_{PWM}}{L_a} \\ 0 \\ 0 \end{pmatrix} u,$$

$$z = \begin{pmatrix} K_T & -B & 0 \end{pmatrix} x + \left(-\frac{1}{J}\right) w,$$

$$u = \omega^{sp} - K_x x, \qquad w = \delta_J z, \quad |\delta_J| \le 1,$$

$$y = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} x,$$

where J_N is the rotor inertia nominal value and δ_J represents the rotor inertia uncertainty such that $J = J_N(1+\delta_J)$, $x \in R^{3x1}$ is the state

$$x = \begin{pmatrix} i_a \\ \omega \\ \int \omega \end{pmatrix}$$

and

$$K_x = \begin{pmatrix} K_p^{i_a} & K_p^{i_a} K_p^{\omega} & K_p^{i_a} K_i^{i_a} \end{pmatrix}.$$

4.2. LMI Conditions

We can readily normalize the system above so that it can be written as [8]:

$$\begin{pmatrix} \frac{dx}{dt} \\ z \\ y \end{pmatrix} = \begin{pmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{pmatrix} \begin{pmatrix} x \\ w \\ u \end{pmatrix}$$

As can be seen in [2], [3], it is possible to formulate the synthesis conditions that ensure specifications (a)÷(d), as defined previously, in a set of *LMI* constraints. The problem is equivalent to $\exists Q > 0$, $T > 0$ and $Y=K_xQ$ such that:

$$\begin{pmatrix} \dfrac{AQ + QA^T + B_uY + Y^TB_u^T + 2\alpha Q}{TB_w^T + C_zQ + D_{yu}Y} \,|\ldots \\ \ldots|\, \dfrac{B_wT + QC_z^T + Y^TD_{yu}^T}{TD_{zw} + D_{zw}^TT - 2T} \end{pmatrix} < 0$$

The *LMI* condition that ensures a bound u_{max} on the command input *u(t)* for every initial condition x_0 in the ellipsoid $\varepsilon_Q = \{x \mid x^TQx \le 1\}$ is:

$$\begin{pmatrix} u_{\max}^2 I & Y \\ \hline Y^T & Q \end{pmatrix} \le 0$$

For every initial conditions that belong to the ellipsoid ε_Q, some bounds $z_{\max}^i$, $i = \overline{1, n_z}$, can also be ensured for outputs $z_i = C_z^i x$ with *LMI* constraints:

$$\left(z_{\max}^i\right)^2 - C_z^i Q \left(C_z^i\right)^T \ge 0 \quad z_{\max}^i, i = \overline{1, n_z}\,.$$

4.3. Simulation

The numerical results are obtained with the *MATLAB*. We synthesized a controller that ensures speed reference tracking for ±5%-variations from its nominal value. We assume a constant load torque of T_L*=0.4 [N.m]* and require a velocity (angular speed) of $\omega_{sp} = 1500[rpm] = 50\pi\left[\frac{rad.}{\sec.}\right]$. This design provides the robustness of the closed-loop system with respect to rotor inertia uncertainties and so that the saturation bounds for current ($i_a \le i_a^{\max} = 10[A]$) and voltage $u_c \le u_c^{\max} = 5[V]$ are not exceeded.

We impose a strong decay-rate by setting $\alpha = 65$ and find the controller $K_p^{i_a} = 0.1411$ $K_p^{\omega} = 0.7514$ $K_i^{\omega} = 45.8043$.

We plot in figure 5 the responses of the closed loop system of angular speed (ω) and current (i_a) when the load inertia varies from *100%* to *200%* of its nominal value.

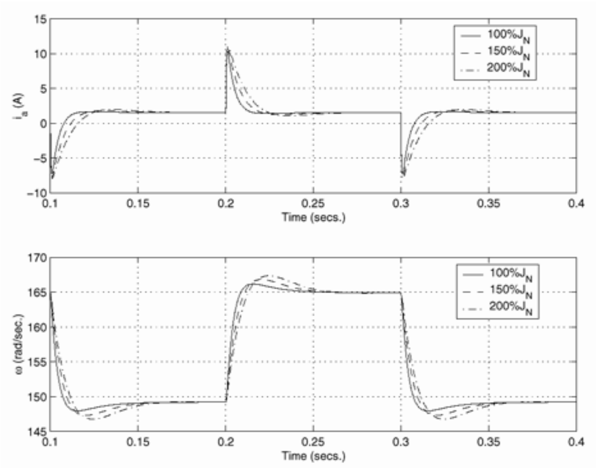

Figure 5. LMI based design: closed loop time response of angular speed (ω) and current (i_a), when load inertia varies from 100% to 200% of the nominal.

5. CONCLUSIONS

Two loop control structures are commonly used in motor drives, in speed applications. In this paper is to quantify the sub optimal of a standard *PI* controller, nested in two-loop control design algorithm for a *DC* motor in speed servo applications with respect to robustness of the system uncertainties. Specifically, we will focus on the case of an uncertain moment of inertia. Our analysis and design tools will be the LMI based methods. These methods are applicable to any motor and uncertainty type and are extendable to positioning problems as well.

The paper re-examines the standard nested two-loop controller structure for a permanent magnet *DC* motor in speed servo applications. A robust synthesis, via *LMI* design, is compared with a conventional pole placed design. An analysis of a generic example demonstrates that robust design offers noticeable improvements in performance only in cases of relatively large model uncertainties.

In this paper, we have extended state-feedback control to systems with parameter uncertainties. The controller presented (via *LMI's*) shows excellent robustness characteristics in comparison with the classical controller design. This work demonstrates that state-feedback robust controller via *LMI's* is very efficient and flexible in practical problems.

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