Controller for Servo DC Motor via Robust Synthesis

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Abstract: In the servomechanism problem, it is desired to design a controller for a plant so that the outputs of the plant are independent, as much as possible, from disturbances which may affect the system (regulation occurs) and also such that the outputs asymptotically track any specified reference input signals applied to the system (tracking occurs), subject to the requirements of maintaining the closed-loop system stability. This paper examines some aspects of controller synthesis for the multivariable servomechanism problem when the plant to be controlled is subject to uncertainty. In this case, a controller is to be designed so that the desired regulation and the tracking take place in spite of the fact that the plant dynamics or/and parameters may vary by arbitrary, large amounts, subject only to the condition that the resultant closed-loop perturbed system remains stable. This problem is called the robust servomechanism problem.

Keywords: DC motor, robust synthesis, Linear Fractional Representation (LFR), $H^\infty$ controller, PI controller.

1. Introduction

Two loop control structures are commonly used in motor drives, in speed applications and a third loop is added if positioning is required. This structure is widely perceived as very satisfactory, both from a control standpoint and from an apparatus protection point of view [6].

The DC motor is provided by a pulse with modulated (PWM) voltage source in a majority of applications. A PWM power supply is an intrinsically nonlinear device due to the switching mode of operation and saturation. At low frequencies the PWM unit is well approximated by a linear gain ($k_{PWM}$), such an approximation being inaccurate at higher frequencies. Control gain roll off requirements are thus included in the conventional design algorithm. In order to make a meaningful comparison between the conventional and the robust controllers we require that control gains have comparable bounds at high frequencies.

2. Conventional controller design

A simplified model of a servo DC motor is presented in [6]. It is assumed that the measurements of the motor speed $\omega$ and the armature current $i_a$ are available and that the control input is the voltage reference signal $V_{ctrl}$. A PI controller $H_i(s)=k_p + \frac{1}{s}$ for the current loop is designed in the first step. Here I ignore the back $E_a(s)=k_c\Omega(s)$, assuming that the current loop response dynamics is much faster. I also ignore the relatively small
effects of $R_{da}$ on $\frac{d}{dt} T_a$. The design of the PI speed controller $H(s) = k_{r_a} + \frac{1}{s} k_{r_a}$, in the second step, a similar procedure is the following: the friction effects are neglected and the much faster current loop is approximated by the identity. Closed loop time responses are computed, when the load inertia varies from 100% to 200% of its nominal value.

3. Standard $H^\infty$ Optimal Control

Consider the standard setup of Figure 1. We must define the concept of internal stability for this setup. Start with a minimal realization of $G$:

$$G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

![Figure 1. The generalized regulator configuration.](image)

The input and output of $G$ are partitioned as:

$$\begin{pmatrix} w \\ u \end{pmatrix}, \begin{pmatrix} z \\ y \end{pmatrix}.$$ 

This induces a corresponding partition of $B$, $C$ and $D$:

$$\begin{pmatrix} B_w & B_u \\ C_z & C_y \end{pmatrix}, \begin{pmatrix} D_{zw} & D_{zu} \\ D_{yw} & D_{yu} \end{pmatrix}.$$

We shall assume that $D_{yu} = 0$, that is, the transfer matrix from $u$ to $y$ is strictly proper. This is a condition to guarantee existence of closed-loop transfer matrices. Thus the realization for $G$ has the form:

$$G(s) = \begin{pmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{pmatrix}$$

Also, bring in a minimal realization of $K$:

$$K(s) = \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix}$$

Now set $w = 0$ and write the state equations describing the controlled system:

$$\begin{pmatrix} \dot{x}_K \\ \dot{x} \end{pmatrix} = \begin{pmatrix} A_K & B_K C_y \\ 0 & A \end{pmatrix} \begin{pmatrix} x_K \\ x \end{pmatrix} + \begin{pmatrix} 0 \\ B_u \end{pmatrix} u$$

$$\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} C_K & D_K C_y \\ 0 & C_y \end{pmatrix} \begin{pmatrix} x_K \\ x \end{pmatrix}$$

Eliminate $u$ and $y$:

$$\begin{pmatrix} \dot{x}_K \\ \dot{x} \end{pmatrix} = \begin{pmatrix} A_K & B_K C_y \\ B_u C_K & A + B_u D_K C_y \end{pmatrix} \begin{pmatrix} x_K \\ x \end{pmatrix}$$

We call this latter matrix the closed-loop $A$-matrix. It can be checked that its eigenvalues do not depend on the particular minimal realizations chosen for $G$ and $K$. The closed-loop system is said to be internally stable if this closed-loop $A$-matrix is stable, that is, all its eigenvalues have negative real part. It can be proved that, given $G$, an internally stabilizing $K$ exists if $(A, B_u)$ is stabilizable and $(C_y, A)$ is detectable.

Let $T_{zw}$ denote the system from $w$ to $z$. The $H^\infty$ optimal control problem is to compute an internally stabilizing controller $K$ that minimizes $\|T_{zw}(s)\|_\infty$ for the standard setup of Figure 1. The following conditions guarantee the existence of an optimal $K$:

(A1) $(A, B_u)$ is stabilizable and $(C_y, A)$ is detectable;

(A2) The matrices $D_{zu}$ and $D_{yw}$ have full column and row rank;

(A3) The matrices:

$$\begin{pmatrix} A - j\omega & B_u \\ C_y & D_{zw} \end{pmatrix}, \begin{pmatrix} A - j\omega & B_w \\ C_y & D_{yw} \end{pmatrix}$$

have full column and row rank, respectively, $\forall \omega$.

Instead of seeking a controller that actually minimizes $\|T_{zw}(s)\|_\infty$, a simpler problem is to search for a controller that gives $\|T_{zw}(s)\|_\infty < \gamma$, where $\gamma$ is a prespecified parameter. If $\gamma$ is too small, a controller will not exist, so we need a test for existence.
With this, the following procedure leads to a controller that is close to optimal:

1. Start with a large enough $\gamma$ so that a controller exists;
2. Test existence for smaller and smaller values of $\gamma$ until eventually $\gamma$ is close to minimum $\gamma$ for existence;
3. Compute a controller so that $\|T_{zw}(s)\|_\infty < \gamma$.

4. A Design Example

4.1. Modeling DC Motor Position

A common actuator in control systems is the DC motor. It directly provides rotary motion and coupled with wheels or drums and cables, can provide transitional motion.

The motor torque, $T$, is related to the armature current, $i_a$, by a constant factor $K_t$.

The back emf, $e$, is related to the rotational velocity by the following equations:

$$
\dot{\vartheta} = \frac{K}{J} i_a - T,
$$

$$
L \frac{di_a}{dt} + R i_a = u_a - K \dot{\vartheta},
$$

where:

- $J$ [kg.m²/s²]: Moment of inertia of the rotor;
- $B$ [N.m.s]: Damping ratio of the mechanical system;
- $K$ = $K_e$ [N.m/A]: Electromotive force constant;
- $R$ [Ω]: Electric resistance;
- $L$ [H]: Electric inductance;
- $u_a$ [V]: Source (armature) voltage;
- $\vartheta$ [rad]: Position of shaft

These equations can also be represented in state-space form. If we choose motor position, motor speed and armature current as our variables, we can write the equations as follows:

$$
\dot{x} = \frac{d}{dt}\begin{pmatrix}
\vartheta \\
\dot{\vartheta} \\
i_a
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 \\
0 & -\frac{B}{J} & \frac{K}{J} \\
0 & \frac{1}{L} & -\frac{K}{L}
\end{pmatrix}
\begin{pmatrix}
\vartheta \\
\dot{\vartheta} \\
i_a
\end{pmatrix} + ...
$$

$$
y = \begin{pmatrix}
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\vartheta \\
\dot{\vartheta} \\
i_a
\end{pmatrix}
$$

We consider the following numerical values: $L=2.75E-6$ H, $R=4$ Ω, $J=3.23E-6$ kg.m²/s², $B=3.51E-6$ N.m.s, $K=0.0274$ N.m/A.

4.2. Performance of Feedback System

The LFR model can be constructed systematically, starting by the system's dynamic equations. It results in a model like the form depicted in Figure 2, where the $G(s)$ represents the “nominal” linear time-invariant system. The matrix $\Delta(J)$ contains rotor inertia uncertainties and is connected to the nominal system via input $w$ and output $z$.

![Figure 2. Linear Fractional Representation for uncertain nonlinear system.](image)

The performance is measured on an augmented plant where $\gamma$ is defined as the upper bound of the $L_2$-induced gain from $w$ to $z$. The plant is augmented by rational weighting functions as shown in Figure 3, according to [7]. The weighting functions are:
Specifically, \( W_{ref} \) defines the designated step response, \( W_{\epsilon} \) is the weight for tracking the step response and the command is penalized by \( W_u \).

### 4.3. Controller Synthesis

The numerical results are obtained with the MATLAB. It is synthesized a controller that ensures speed reference tracking for \( \pm 0.5\% \)-variations from its nominal value. This design provides the robustness of the closed-loop system with respect to the saturation bounds for current and voltage are not exceeded. The step response of closed loop system with standard \( H^\infty \) controller is shown in Figure 4.

## 5. Conclusions

The robust servomechanism problem consists in finding an LTI controller so that:

(i) the resultant closed-loop system is asymptotically stable,

(ii) asymptotic tracking occurs, and

(iii) condition (ii) holds for any arbitrary perturbations in the plant model (parametric uncertainty or dynamic uncertainty, including changes in model order) that do not cause the resultant closed-loop system to become unstable.

It is synthesized a controller that ensures speed reference tracking for \( \pm 5\% \)-variations from its nominal value. This design provide the robustness of the closed-loop system with respect to rotor inertia uncertainties and so that the saturation bounds for current and voltage are not exceeded.

## REFERENCES


