Discrete Hilbert Transform. Numeric Algorithms

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Abstract - *The Hilbert and Fourier transforms are tools used for signal analysis in the time/frequency domains. The Hilbert transform is applied to casual continuous signals. The majority of the practical signals are discrete signals and they are limited in time. It appeared therefore the need to create numeric algorithms for the Hilbert transform. Such an algorithm is a numeric operator, named the Discrete Hilbert Transform. This paper makes a brief presentation of known algorithms and proposes an algorithm derived from the properties of the analytic complex signal. The methods for time and frequency calculus are also presented.*

1. INTRODUCTION

Signals can be classified into two classes: analytic signals (for instance $x(t) = A\sin\omega t$), and experimental signals (measured signals). The last category represents real signals and is of great importance in applications.

An experimental signal represents a signal observed during a limited interval of time. It is a sample of the original signal, which characterizes a physical process of interest.

The experimental signal can be a continuous time signal (analogical), or a digital signal (discrete).

The practical limitations of the systems used to analyze analogical signals impose that the experimental analogical signals had a limited frequency band [1],[2].

If the original signal doesn't have a limited band, a low-pass filtration needs to be applied in order to obtain the experimental signal which will be analyzed.

The rule also applies to sampled signals, which need to have a limited frequency band too.

As a result, before acquisition, the experimental analogical signal will be low-pass filtered.

The acquisition frequency needs to be two times the biggest frequency of the signal's spectrum in order to avoid the aliasing process – the Nyquist condition.

The discrete signal will be analyzed on a computer system, which implies its digitization (the digital signal is the discrete signal converted in binary format, accordingly to the adopted analog/numeric conversion; in most of the cases, the signal acquisition hardware also does the digitization of the signal samples). The resulted digital signal has the greatest importance in numeric analysis operations.

Some other remarks need to be made. Since the sampled signal has a limited length, it needs to either (1) have a infinite frequency spectrum, or (2) be a periodic signal. In case (1), the sampling doesn't respect the Nyquist condition. Or, in case (2) we choose to represent the signal as a periodic one, with an extended period. In both cases, the digital signal cannot exactly represent the original physical process.

In the case of the Hilbert transform, it's a known fact that the signal *x(t)* needs to be causal (that is *x(t)=0*, for $t < 0$). The sampled signal *x[n]* is in this case a non-periodic sequence, real and causal.

In such a case, we can talk of a discrete Hilbert transform applied to the sequence *x[n]*.

The complex analytic signal associated to the *x[n]* sequence has the spectrum different

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from zero only for the interval of positive frequencies.

When *x(t)* is a periodic signal, *x[n]* is a periodic sequence and we cannot talk of causality (the periodic term implies the sequence extension from $-\infty$ to $+\infty$). A calculus algorithm for the Discrete Hilbert Transform in this case imposes the condition that the Discrete Fourier Transform of the complex analytic sequence to be equal to zero in the interval of negative frequencies. And of course, for positive frequencies, the spectrum of the analytic sequence to be two times the spectrum of the signal *x[n]*. In this case, the Hilbert transform can be used with all its known advantages regarding the causal signals.

The next paragraphs present the methods for calculating the Discrete Hilbert Transform.

2. HILBERT TRANSFORM IN CONTINUOUS TIME

To start, we present first the theory of the Hilbert transform, definitions, properties [2], [10].

Let's consider a real measurement signal:

$$x(t) \in \boldsymbol{L}^{(2)} \quad (1)$$

Where $\mathbf{L}^{(2)}$ is the signal class with integral square.

The Hilbert transform of the signal *x(t)* is:

$$\hat{x}(t) = \boldsymbol{H}\{x(t)\} = \frac{1}{\pi} v \cdot p \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau \quad (2)$$

$$\text{v.p.} \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} dt = \lim_{\varepsilon \to 0} \left[\int_{-\infty}^{t-\varepsilon} \frac{x(\tau)}{t-\tau} d\tau + \int_{t+\varepsilon}^{\infty} \frac{x(\tau)}{t-\tau} d\tau \right],$$

where v.p. represents the functional named principal value.

$\hat{x}(t)$ is improper named the conjugate of *x(t)*.

We also have $\hat{x}(t) \in \boldsymbol{L}^{(2)}$.

x(t) is the inverse Hilbert transform of $\hat{x}(t)$:

$$x(t) = \boldsymbol{H}^{-1}\{\hat{x}(t)\} = -\frac{1}{\pi} v.p. \int_{-\infty}^{\infty} \frac{\hat{x}(t)}{t-\tau} d\tau \quad (3)$$

Let's observe that $\hat{x}(t)$ is determined by the convolution of *x(t)* with the signal $\frac{1}{\pi t}$:

$$\hat{x}(t) = x(t) * \frac{1}{\pi t}. \quad (4)$$

The above relation allows the calculus of the spectral density of $\hat{x}(t)$:

$$\hat{X}(j\omega) = \boldsymbol{F}\{\hat{x}(t)\} = \boldsymbol{F}\{x(t)\} \cdot \boldsymbol{F}\left\{\frac{1}{\pi t}\right\} \quad (5)$$

Or

$$\hat{X}(j\omega) = X(j\omega) \cdot \boldsymbol{F}\left\{\frac{1}{\pi t}\right\} \quad (6)$$

Since:

$$\boldsymbol{F}\left\{\frac{1}{\pi t}\right\} = -j \operatorname{sgn}(\omega)$$

It results:

$$\hat{X}(j\omega) = X(j\omega)[-j \operatorname{sgn} \omega] \quad (7)$$

Or:

$$\hat{X}(j\omega) = \begin{cases} -j\, X(j\omega), & \omega > 0 \\ j\, X(j\omega), & \omega < 0 \end{cases} \quad (8)$$

As a result, the spectral density function of the *x(t)* signal's conjugate is obtained by changing the phase of the spectral density for $X(j\omega)$ by $\pm \pi/2$.

It results:

$$\boldsymbol{H}\{x(t)\} = \hat{x}(t) = \boldsymbol{F}^{-1}\{\hat{X}(j\omega)\} \quad (9)$$

The inverse Hilbert transform is defined in relation (3). We can write:

$$x(t) = \boldsymbol{H}^{-1}\{\hat{x}(t)\} = -\boldsymbol{H}\{\hat{x}(t)\} \quad (10)$$

Taking into account relation (8) it results:

$$x(t) = -H\{\hat{x}(t)\} = \begin{cases} F^{-1}\{j\hat{X}(j\omega j\}, \omega > 0 \\ F^{-1}\{-j\hat{X}(j\omega j\}, \omega < 0 \end{cases} \quad (11)$$

The analytic signal

Having the pairs x (t) and $\hat{x}(t) = \boldsymbol{H}\{x(t)\}$ we build the analytic signal z (t) :

$$z(t) = x(t) + j\,\hat{x}(t) \quad (12)$$

We observe that:

$$Z(j\omega) = F\{z(t)\} = X(j\omega) + j\,\hat{X}(j\omega) \quad (13)$$

Referring to relation (7) we obtain:

$$Z(j\omega) = X(j\omega) + j[-j\,\text{sgn}\,\omega]X(j\omega) = \\ = X(j\omega)[1+\text{sgn}\,\omega] = 2X(j\omega)u(\omega) \quad (13.a)$$

where $u(\omega)$ is the unit step function.

It's useful to observe that:

$$X(j\omega) = \frac{1}{2}[Z(j\omega) + Z^*(-j\omega)] \quad (14)$$

$$\hat{X}(j\omega) = \frac{1}{2j}\left[Z(j\omega) - Z^*(-j\omega)\right] \quad (15)$$

3. DISCRETE HILBERT TRANSFORM. CALCULUS ALGORITHMS.

Definitions

Having the signal *x(t)* defined on the time interval $[0, t_N]$, using a sampling period T_e, we obtain the discrete signal *x[n]* :

$$x[n] = x(nT_e),\ n \in \overline{0, N-1} \quad (16)$$

Where: $T_e = \frac{t_N}{N}$.

The sampling frequency f_e is chosen so that the frequency $\frac{f_e}{2}$ is greater or equal to the least significant frequency from the spectrum of *x(t).* We consider the discrete frequency step $f_0 = \frac{f_e}{N}$, $\omega_0 = \frac{2\pi}{N} f_e$ respectively.

The discrete Fourier transform (DFT) is:

$$TFD\{x([n]\} = X[k] = \sum_{n=0}^{N-1} x[n]e^{-jnk\frac{2\pi}{N}},\ k \in \overline{0, N-1} \quad (17)$$

And the inverse discrete Fourier transform DFT^{-1} is:

$$TFD^{-1}\{X[[k]\} = x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{jnk\frac{2\pi}{N}},\ k \in \overline{0, N-1} \quad (18)$$

The sample of the spectral density corresponding to frequency $k\omega_0$ is determined with the relation:

$$X(jk\omega_0) = T_e X[k]$$

where *X(jω)* is the Fourier transform in continuous time.

On the other hand:

$$X[k]^* = X[N-k] = X[-k]. \quad (19)$$

Which shows that the sample $X[N-k] = X[-k]$ has a correspondent sample of the spectral density, with the negative frequency $X(-k\omega_0)$.

For N – even, the samples

$$\left\{X[1], X[2], ..., x\left[\frac{N}{2}-1\right]\right\}$$

are named „positive harmonics", while the samples:

$$\left\{X[\frac{N}{2}+1], X[\frac{N}{2}+2], ..., X[N-2], X[N-1]\right\} \equiv \\ \equiv \left\{X\left[-\left(\frac{N}{2}-1\right)\right], X\left[-\left(\frac{N}{2}-2\right)\right], ..., X[-2], X[-1]\right\}$$

are named „negative harmonics".

For N – odd, the samples

$$\left\{X[1], X[2], ..., x\left[\frac{N-1}{2}\right]\right\}$$

are called „positive harmonics", while the samples:

$$\left\{X[\frac{N+1}{2}], X[\frac{N+1}{2}+1], ..., X[N-2], X[N-1]\right\} \equiv \\ \equiv \left\{X\left[-\left(\frac{N-1}{2}\right)\right], X\left[-\left(\frac{N-1}{2}-1\right)\right], ..., X[-2], X[-1]\right\}$$

are called „negative harmonics".

The X[0] component is the continuous component, while the $X\left[\frac{N}{2}\right]$ (N - even) is the Nyquist component. It is found when the number of samples, N, is even – a situation frequently found because DFT is implemented using an algorithm for which N is even. Also, The $X\left[\frac{N}{2}\right]$ component is the continuous one.

Similarly to relation (10), the discrete Hilbert transform is defined:

$$H\{x[n]\} = \hat{x}[n] = TFD^{-1}\{\hat{X}[k]\} \quad (20)$$

Where for N – even:

$$\hat{X}[k] = \begin{cases} -jX[k], & k = \overline{1, \frac{N}{2}-1}, N\ even \\ jX[k], & k = \overline{\frac{N}{2}+1, N-1}, N\ even \end{cases} \quad (21a)$$

We observe that the continuous and Nyquist components are excluded (for k = 0 and $k = \frac{N}{2}$).

While for N - odd:

$$\hat{X}[k] = \begin{cases} -jX[k], & k = \overline{1, \frac{N-1}{2}}, N\ odd \\ jX[k], & k = \overline{\frac{N+1}{2}, N-1}, N\ odd \end{cases} \quad (21b)$$

Where the continuous component is excluded.

Calculus Algorithms

Relation (20) can be written:

$$\hat{x}[n] = H\{x[n]\} = TFD^{-1}\{-jS[k]X[k]\}$$

(22)

Where:

$$S[k] = \begin{cases} 1, & k = \overline{1, \frac{N}{2} - 1}, N\ even \\ 0, & k = 0, \quad k = \frac{N}{2} \quad N\ even \\ -1 & k = \overline{\frac{N}{2} + 1, \quad N-1}, N\ even \end{cases} \quad (23a)$$

And for N - even

$$S[k] = \begin{cases} 1, & k = \overline{1, \frac{N-1}{2}}, N\ odd \\ 0, & k = 0, \quad N\ odd \\ -1 & k = \overline{\frac{N+1}{2}, \quad N-1}, N\ odd \end{cases} \quad (23b)$$

S[k] is a window which filters the interest components of the Hilbert transform.

Observations:

The sequence $S[k]$ can be obtained:

1) Using the function:

$$S[k] = \operatorname{sgn}\left\{\sin k\frac{2\pi}{N}\right\}; \ k = \overline{0, N-1} \quad (24a)$$

2) Or the function [10]:

$$S[k] = sgn[k]sgn[\frac{N}{2} - 2] \quad (24b)$$

The following methods of calculating the discrete Hilbert transform result:

3.1. The inverse discrete Fourier transform algorithm

Is based on relations (20) – (21.a) :

1) We determine the discrete Fourier transform of the numeric sequence x [n]:

   $X[k] = TFD\{x[n]\}$

2) We set the continuous component to zero:

   X[0]=0

3) If the length N of sequence *X[k]* is even, we set the Nyquist component to zero:

   $X\left[\frac{N}{2}\right] = 0$

4) The sequences *X[k],* $k \in \overline{1, \frac{N}{2} - 1}, \ N\, even; or \ k = \overline{1, \frac{N-1}{2}}, \ N \ odd.$ (positive harmonics) are multiplied by –j.

5) The sequences $X[k], k \in \overline{\frac{N}{2}+1, N-1}, N even, or \ k = \overline{1, \frac{N+1}{2}, N-1}, \ N\ odd$ (negative harmonics) are multiplied by +j.

6) We calculate the discrete Hilbert transform using relation (20) .

3.2. Windowing the positive and negative frequencies algorithm

Is obtained by applying a window to the positive and negative spectral components, except for the continuous and the Nyquist components. This algorithm is based on relations (22) – (24):

1) N is chosen, the number of sampling points.

2) S[k] is determined, using relation (24.b).

3) The discrete Fourier transform of the numeric sequence *x[n]* is calculated:

   $X[k] = TFD\{x[n]\}$

4) We determine

   $\hat{X}[k] = -jS[k]X[k]$

5) And then we determine

   $\hat{x}[n] = H\{x[n]\} = TFD^{-1}\{-jS[k]X[k]\}$

This algorithm is, at first view, similar with the previous one, except for the last two steps.

3.3. The convolution algorithm

1) We determine

$$s[n] = TFD^{-1}\{-jS[k]\} \quad (25)$$

2) It results:

$$\hat{x}[n] = x[n] \otimes s[n] \quad (26)$$

That is:

$$\hat{x}[n] = \sum_{m=0}^{N-1} x[m]s[n-m] \quad (27)$$

In [10], *s[n]* has this expression:

$$s[n] = \frac{2}{N}\sin^2\left(\frac{\pi n}{2}\right)\cot\left(\frac{\pi n}{N}\right), \text{ for N even } (28)$$

And *s[n]* equal to zero for n= 0, 2,4,... .
While

$$s[n] = \frac{1}{N}\left(\cot\left(\frac{\pi n}{N}\right) - \frac{\cos(\pi n)}{\sin(\frac{\pi n}{N})}\right) \text{ for N odd } (29)$$

And *s[n]* doesn't become equal to zero for n even or odd. Let's note that $s[N-k] = -s[n], \quad n = \overline{1, N-1}$.

This algorithm seems to be computed in a shorter time. In fact, it requires a longer time than the algorithms that use the discrete Fourier transform. This is explained by the fact that for DFT were developed fast calculus algorithms (FFT -Fast Fourier Transform).

Observation

In literature we meet the relation:

$$\tilde{s}[n] = TFD^{-1}\{S[k]\} \quad (30)$$

And then $\hat{x}[n]$ is calculated using this relation:

$$\hat{x}[n] = -j\sum_{m=0}^{N-1} x[m]\tilde{s}[n-m] = -jx[n] \otimes \tilde{s}[n] \quad (31.a)$$

3.4. Windowing the positive frequencies algorithm

We now propose a new algorithm for calculating the discrete Hilbert transform.

Similarly to relation (12) the discrete complex analytic signal is defined (also called complex analytic sequence):

$$z[n] = x[n] + j\,\hat{x}[n] \quad (31)$$

The discrete Fourier transform of the z[n] signal is:

$$Z[k] = TFD\{z[n]\} \quad (32)$$

Looking at (13 c) it can be written:

$$Z[k] = 2TFD\{x[n]\} = 2X[k], \quad k = \overline{1, \frac{N}{2} - 1}, \quad \text{N even}$$

or:

$$k = \overline{1, \frac{N-1}{2}}, \quad \text{N odd} \quad (32\text{ a})$$

The window sequence R[k] is introduced:

$$R[k] = \begin{cases} 0, & k = 0 \\ 2, & k = \overline{1, \frac{N}{2} - 1} \\ 0, & k = \overline{\frac{N}{2}, N-1} \end{cases}, \quad N \text{ even} \quad (33\text{ a})$$

Or:

$$R[k] = \begin{cases} 0, & k = 0 \\ 2, & k = \overline{1, \frac{N-1}{2}} \\ 0, & k = \overline{\frac{N+1}{2}, N-1} \end{cases}, \quad N \text{ odd} \quad (33\text{ b})$$

It can be observed that:

$$R[k] = S[k](S[k]+1) \quad (33\text{ c})$$

In a more compact form it can be written:

$$Z[k] = R[k]X[k], \quad k = \overline{1, N-1} \quad (34)$$

This relation results:

$$z[n] = TFD^{-1}\{R[k]X[k]\} \quad (35)$$

Where DFT^{-1} has the meaning of the inverse complex Fourier transform. It results that:

$$x[n] = \text{Re}\{z[n]\} \quad (36)$$

And:

$$\hat{x}[n] = THD\{x[n]\} = \text{Im}\{z[n]\} \quad (37)$$

Relations (31), (37) are leading to an algorithm which allows calculating the discrete Hilbert transform.

Observation: The relation (35) can also be written as:

$$z[n] = \frac{1}{N}\sum_{k=1}^{M} Z[k] e^{jnk\frac{2\pi}{N}}, \quad M = \frac{N}{2} - 1, \quad \text{for } N \text{ even and } M = \frac{N-1}{2}, \text{ for } N \text{ odd}. \tag{38}$$

The relation (38) doesn't have a practical use since the numeric signal analysis environments already have instruments for computing the DFT – (FFT). As a result, it is preferred the algorithm expressed by the relations (31) – (32).

4. INVERSE DISCRETE HILBERT TRANSFORM

Is determined using the relation:

$$\mathbf{H}^{-1}\{\hat{x}[n]\} = -\mathbf{H}\{\hat{x}[n]\} \tag{39}$$

If *N*, the sequence length is odd and the continuous component is missing we can write:

$$\mathbf{H}^{-1}\{\hat{x}[n]\} = x[n] \tag{40}$$

If N is even, while the continuous component is different from zero or if N is odd, that is we have the Nyquist component, then relation (40) is not strictly true any more.

Algorithms of the inverse discrete Hilbert transform similar to those presented above for the direct discrete Hilbert transform result.

5. RESULTS AND CONCLUSIONS

This paper briefly presents known algorithms for calculating the Hilbert transform and proposes and algorithm based on the properties of the complex analytic signal. The methods of computation in time and frequency domains are presented. For applications where the hard/soft throughput as well as time are important issues, this algorithm could represent an advantage.

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